The Triangular Distribution

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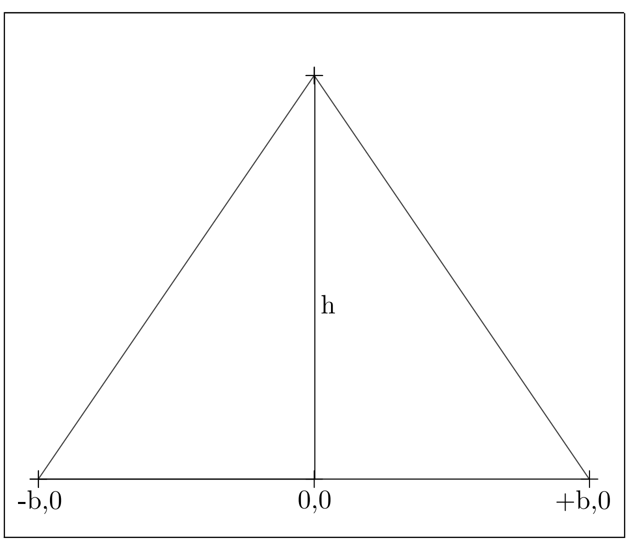
Often times we need a distribution for a random variable but we don't have any firm convictions as to the shape of the probability distribution but we know it's endpoints and we know that the probability distribution is not uniform (i.e. the end points are less probable than the mean). One distribution that we might want to consider is the Triangular Distribution. Some advantages of this distribution are (1) the math for the probability density function is easy, (2) the domain of x is restricted to the interval [a,b] and (3) the distribution is defined solely by its first two moments (i.e. the distribution is symmetric). We will use this distribution to solve the following problem...

The Problem

We have a company whose revenues are projected to be \$50 million in some future year t. Current annualized revenue is \$10 million. We can say with some conviction that gross revenue in year t will not be less than \$10 million and will not be greater than \$90 million. Using a triangular distribution what are the probabilities that (1) revenue in year t will be less than \$30 million and (2) revenue in year t will be greater than \$60 million?

The Math

We will define the triangular distribution as an isosceles triangle. An isosceles triangle is a triangle with at least two equal sides. The base of the triangle will be centered at the origin and has length of "2b". The triangle has height equal to "h". The graph of this triangle is...



We will divide the isosceles triangle above into two right triangles of equal areas with the left triangle having the x:y coordinates [0,0]:[0,h]:[-b,0] and the right triangle having the x:y coordinates [0,0]:[0,h]:[+b,0]. The equations for the height of the left triangle [f(x)] at any point along the x axis between -b and zero and the height of the right triangle [g(x)] at any point along the x axis between zero and +b are...

$$f(x) = \frac{h}{b}\,x + h \tag{1}$$
$$g(x) = -\frac{h}{b}\,x + h \tag{2}$$

The equation for the area of left triangle (A_L) is...

$$\begin{aligned}
A_L &= \int_{-b}^{0} f(x)\delta x \\
&= \int_{-b}^{0} \left[\frac{h}{b}\,x + h\right]\delta x \\
&= \frac{h}{b}\int_{-b}^{0} x\,\delta x + h\int_{-b}^{0}\delta x \\
&= \frac{h}{b}\frac{1}{2}x^2\Big[_{-b}^{0} + h\,x\Big[_{-b}^{0} \\
&= \frac{h}{b}\frac{1}{2}\left\{0^2 - (-b)^2\right\} + h\left\{0 - (-b)\right\} \\
&= -\frac{1}{2}hb + hb \\
&= \frac{1}{2}hb
\end{aligned} \tag{3}$$

The equation for the area of right triangle (A_R) is...

$$\begin{aligned}
A_R &= \int_{0}^{b} g(x)\delta x \\
&= A_L \\
&= \frac{1}{2}hb
\end{aligned} \tag{4}$$

To convert the area of the triangle into a probability distribution we will need the variable theta such that...

$$\int_{-b}^{0} \theta f(x)\delta x = 0.5000 \;\; ...and... \;\; \int_{0}^{+b} \theta g(x)\delta x = 0.5000 \tag{5}$$

The equation for theta is therefore...

$$\begin{aligned}
\int_{-b}^{0} \theta f(x)\delta x &= 0.5000 \\
\theta\int_{-b}^{0} f(x)\delta x &= 0.5000 \\
\theta\frac{1}{2}hb &= 0.5000 \\
\theta &= \frac{1}{hb}
\end{aligned} \tag{6}$$

The probability that $x \leq z$ where $z \leq 0$ is therefore...

$$\begin{aligned}
P\Big[x \leq z\Big] &= \int\limits_{-b}^{z} \theta f(x) \delta x \\
&= \int\limits_{-b}^{z} \frac{1}{hb}\Big[\frac{h}{b}\,x + h\Big]\delta x \\
&= \frac{1}{b^2}\int\limits_{-b}^{z} x\,\delta x + \frac{1}{b}\int\limits_{-b}^{z} \delta x \\
&= \frac{1}{2}\frac{1}{b^2}x^2\Big[_{-b}^{z} + \frac{1}{b}x\Big[_{-b}^{z} \\
&= \frac{1}{2}\frac{1}{b^2}\Big\{z^2 - (-b)^2\Big\} + \frac{1}{b}\Big\{z - (-b)\Big\} \\
&= \frac{1}{2}\frac{1}{b^2}\Big\{z^2 - b^2\Big\} + \frac{1}{b}\Big\{z + b\Big\} \\
&= \frac{z^2}{2b^2} - \frac{1}{2} + \frac{z}{b} + 1 \\
&= \frac{1}{2} + \frac{z^2}{2b^2} + \frac{z}{b} \qquad (7)
\end{aligned}$$

The probability that $x \leq z$ where $z \geq 0$ is therefore...

$$\begin{aligned}
P\Big[x \leq z\Big] &= \frac{1}{2} + \int\limits_{0}^{z} \theta\, g(x)\, \delta x \\
&= \frac{1}{2} + \int\limits_{0}^{z} \frac{1}{hb}\Big[-\frac{h}{b}\,x + h\Big]\delta x \\
&= \frac{1}{2} - \frac{1}{b^2}\int\limits_{0}^{z} x\,\delta x + \frac{1}{b}\int\limits_{0}^{z} \delta x \\
&= \frac{1}{2} - \frac{1}{2}\frac{1}{b^2}\Big\{z^2 - 0^2\Big\} + \frac{1}{b}\Big\{z - 0\Big\} \\
&= \frac{1}{2} - \frac{z^2}{2b^2} + \frac{z}{b} \qquad (8)
\end{aligned}$$

The Problem Solution

The first thing that we have to do is map the revenue range to the triangle base. To do this we need the value of the "b" parameter. If the variable X_U is the upper revenue range of 90,000 and the variable X_L is the lower revenue range of 10,000 the value of b is...

$$\begin{aligned}
b &= \frac{1}{2}\Big[X_U - X_L\Big] \\
&= \frac{1}{2}\Big[90,000 - 10,000\Big] \\
&= 40,000 \qquad (9)
\end{aligned}$$

The next thing that we have to do is map the and 30,000 threshold (x) in the problem above to the variable z which is...

$$
\begin{aligned}
z &= -b + \Big[x - X_L\Big] \\
&= -40,000 + \Big[30,000 - 10,000\Big] \\
&= -20,000
\end{aligned} \tag{10}
$$

The answer to the first part of the problem is...

$$
\begin{aligned}
P[revenue \leq 30,000] &= \frac{1}{2} + \frac{z^2}{2b^2} + \frac{z}{b} \\
&= \frac{1}{2} + \frac{-20,000^2}{2 \times 40,000^2} + \frac{-20,000}{40,000} \\
&= 0.500 + 0.125 - 0.500 \\
&= 0.125
\end{aligned} \tag{11}
$$

We then map the 60,000 threshold (x) in the problem above to the variable z which is...

$$
\begin{aligned}
z &= -b + \Big[x - X_L\Big] \\
&= -40,000 + \Big[630,000 - 10,000\Big] \\
&= 10,000
\end{aligned} \tag{12}
$$

The answer to the second part of the problem is...

$$
\begin{aligned}
P[revenue \geq 60,000] &= 1.00 - P[revenue \leq 60,000] \\
&= 1.00 - \left[\frac{1}{2} - \frac{z^2}{2b^2} + \frac{z}{b}\right] \\
&= 1.00 - \left[\frac{1}{2} - \frac{10,000^2}{2 \times 40,000^2} + \frac{10,000}{40,000}\right] \\
&= 1.00 - \Big[0.500 - 0.03125 + 0.25\Big] \\
&= 0.28125
\end{aligned} \tag{13}
$$

To summarize the probability that revenue will be less than \$30 million and greater than \$60 million are 0.125 and 0.28125, respectively.